

# Characterization of Monge-Ampère measures with Hölder continuous potentials

Tien-Cuong Dinh and Viêt-Anh Nguyễn

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## Abstract

We show that the complex Monge-Ampère equation on a compact Kähler manifold  $(X, \omega)$  of dimension  $n$  admits a Hölder continuous  $\omega$ -psh solution if and only if its right-hand side is a positive measure with Hölder continuous super-potential. This property is true in particular when the measure has locally Hölder continuous potentials or when it belongs to the Sobolev space  $W^{2n/p-2+\epsilon, p}(X)$  or to the Besov space  $B_{\infty, \infty}^{\epsilon-2}(X)$  for some  $\epsilon > 0$  and  $p > 1$ .

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## 1 Introduction

Let  $(X, \omega)$  be a compact Kähler manifold of dimension  $n$ . Recall that a function  $u : X \rightarrow \mathbb{R} \cup \{-\infty\}$  is said to be  $\omega$ -psh if it is locally the difference of a psh function and a potential of  $\omega$ , i.e. a smooth function  $v$  such that  $dd^c v = \omega$ . We consider the complex Monge-Ampère equation

$$(dd^c \varphi + \omega)^n = \mu,$$

where  $\mu$  is a positive measure on  $X$  and  $\varphi$  a bounded  $\omega$ -psh function, see Bedford-Taylor [2], Demailly [6] and Fornæss-Sibony [21] for the intersection of currents and for basic properties of psh functions. For cohomology reason, the above relation implies that the mass of  $\mu$  is equal to the mass of the measure  $\omega^n$ , i.e.

$$\|\mu\| = \int_X \omega^n.$$

In what follows, we always assume this condition.

When  $\mu$  is a smooth volume form, the famous theorem of Calabi-Yau says that the Monge-Ampère equation admits a smooth solution  $\varphi$  which is unique up to an additive constant [5, 35]. In this paper, we consider the case with Hölder continuous solutions. Without reviewing the long history of the complex Monge-Ampère equation, let us mention few steps in the recent development.

In [26, 27], Kolodziej has constructed a continuous solution under some hypothesis on the measure  $\mu$ , in particular for  $\mu$  of class  $L^p$ ,  $p > 1$ . Then, he proved in [28] that the solution is Hölder continuous when  $\mu$  is of class  $L^p$ ,  $p > 1$ , see also [20, 22]. Some important steps in his approach were improved by Dinew and Zhang [12]. Very recently based on Demailly's regularization method [7, 8] and the above Dinew-Zhang's results, Demailly, Dinew, Guedj, Hiep, Kolodziej, Zeriahi [9] obtained an explicit Hölder exponent of the solution, see also [11]. A necessary condition on  $\mu$  to have a Hölder continuous solution  $u$  was obtained by Dinh-Nguyen-Sibony in [13]. We refer to the works of the above mentioned authors and Eyssidieux, Pali, Plis, Song, Tian [9, 10, 11, 12, 13, 20, 25, 26, 27, 28, 29, 32, 33] for results in this direction, for related topics and a more complete list of references.

Here is our main theorem which implies several known results. The proof uses the above results and some ideas from the works by Sibony and the authors [14, 17, 18, 19].

**Theorem 1.1.** *Let  $(X, \omega)$  be a compact Kähler manifold of dimension  $n$ . Let  $\mu$  be a positive measure on  $X$  of mass  $\int_X \omega^n$ . Then the Monge-Ampère equation*

$$(dd^c \varphi + \omega)^n = \mu$$

*admits a Hölder continuous  $\omega$ -psh solution  $\varphi$  if and only if  $\mu$  admits a Hölder continuous super-potential.*

The Hölder exponent of the solution depends on  $n$  and on the Hölder exponent of the super-potential  $\mathcal{U}$  of  $\mu$ . It will be specified in the proof of Theorem 1.1 in Section 3.

We will recall the notion of super-potentials for measures in Section 3. Super-potentials for positive closed currents were introduced by Sibony and the first author in [17, 18, 19]. They play an important role in complex dynamics. In the references above, situations where the Hölder continuity of super-potential can be verified, are described. The reader will also find in Section 3 some methods to obtain this property. In particular, we will see that the class of measures with Hölder continuous super-potential contains the measures given by  $L^p$  forms,  $p > 1$ , considered in [9, 12, 28] and the measures satisfying some Hausdorff type regularity investigated in [25].

The following corollary is a consequence of Theorem 1.1 and Proposition 3.7 below.

**Corollary 1.2.** *Under the hypotheses of Theorem 1.1, if we can write locally  $\mu = dd^c U + \partial V + \bar{\partial} W$  with Hölder continuous forms  $U, V, W$  of bidegrees  $(n-1, n-1)$ ,  $(n-1, n)$  and  $(n, n-1)$  respectively, then the considered Monge-Ampère equation admits a Hölder continuous  $\omega$ -psh solution. Moreover, the hypothesis on  $\mu$  is satisfied when this measure belongs to the Sobolev space  $W^{2n/p-2+\epsilon, p}(X)$  or to the Besov space  $B_{\infty, \infty}^{\epsilon-2}(X)$  for some  $\epsilon > 0$  and  $p > 1$ .*

In the case with parameters  $(X_t, \omega_t, \mu_t)$ , where the compact Kähler manifolds  $(X_t, \omega_t)$  have uniformly bounded geometry and the super-potentials of the measures  $\mu_t$  are uniformly Hölder continuous, we obtain solutions  $\varphi_t$  which are uniformly Hölder continuous. We can also extend our results to the case of a big and nef class with a solution locally Hölder continuous on the ample locus. We refer to [9, 12, 20] for the techniques needed to handle these situations.

The following problem suggested by the work of Sibony and the authors [13] is still open. We refer to [9, 14] for some particular cases where the answer is positive.

**Problem 1.3.** *Let  $\mu$  be a probability measure on  $X$ . Assume that  $\mu$  is moderate. Does the Monge-Ampère equation*

$$(dd^c \varphi + \omega)^n = \mu$$

*admit a Hölder continuous  $\omega$ -psh solution  $\varphi$  ?*

The notion of moderate measures will be recalled in Section 2. We think that the answer is not always positive and the problem requires probably a better understanding of the notion of capacity  $\mathcal{T}(\cdot)$ , see [16] and Section 2 for the definition.

**Problem 1.4.** *Characterize the positive measures  $\mu$  on  $X$  such that the associated complex Monge-Ampère equation admits a continuous (resp. bounded) solution.*

Note that it is not difficult to show using [6, Ch.III (3.6) and (3.11)] that when  $\varphi$  is a continuous (resp. bounded)  $\omega$ -psh function the measure  $(dd^c \varphi + \omega)^n$  has a continuous (resp. bounded) super-potential. However, in dimension  $n \geq 2$ , the last property does not characterize the Monge-Ampère measures with continuous (resp. bounded) potential. We can consider, for example in dimension 2, a measure with a single singularity likes  $(dd^c \log(-\log \|z\|))^2$ . It has a continuous super-potential.

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## 2 Moderate measures and capacities

In what follows,  $(X, \omega)$  always denotes a compact Kähler manifold of dimension  $n$ . Recall from [13, 14] that a positive measure  $\mu$  on  $X$  is *moderate* if there are constants  $c > 0$  and  $\alpha > 0$  such that

$$\int e^{-\alpha u} d\mu \leq c \quad \text{for every } u \text{ } \omega\text{-psh such that } \int_X u \omega^n = 0.$$

Note that the functions  $u$  satisfying the last condition describe a compact set of  $\omega$ -psh functions. The condition can be replaced by other ones, e.g.  $\max_X u = 0$ .

**Lemma 2.1.** *Let  $\mu$  be a moderate positive measure on  $X$ . Then there are constants  $c > 0$  and  $\alpha > 0$  such that for any  $M \geq 0$*

$$\mu\{u < -M\} \leq ce^{-\alpha M} \quad \text{for every } \omega\text{-psh function } u \text{ such that } \int_X u \omega^n = 0.$$

Moreover, if  $p \geq 1$  is a real number, then there is a constant  $c_p > 0$  such that

$$\|u\|_{L^p(\mu)} \leq c_p \quad \text{and} \quad \int_{\{u < -M\}} |u|^p d\mu \leq c_p e^{-\alpha M} \quad \text{for all } M \text{ and } u \text{ as above.}$$

*Proof.* The first assertion follows from the definition of moderate measure. Observe that  $u$  is bounded above by a constant independent of  $u$ . Therefore, the second assertion in the lemma follows from the first one and the identities

$$\int_{|u| > M} (|u|^p - M^p) d\mu = \int_{M^p}^{\infty} \mu\{|u|^p > r\} dr = p \int_M^{\infty} \mu\{|u| > t\} t^{p-1} dt$$

for  $M \geq 0$ . We change  $\alpha$  if necessary.  $\square$

**Lemma 2.2.** *Let  $\mu$  be a moderate positive measure on  $X$ . Then for any real number  $p \geq 1$ , there is a constant  $c > 0$  such that*

$$\|u - u'\|_{L^p(\mu)} \leq c \max(1, -\log \|u - u'\|_{L^1(\mu)})^{(p-1)/p} \|u - u'\|_{L^1(\mu)}^{1/p}$$

for all  $\omega$ -psh functions  $u$  and  $u'$  such that  $\int_X u \omega^n = \int_X u' \omega^n = 0$ .

*Proof.* In what follows,  $\lesssim$  and  $\gtrsim$  denote inequalities up to a multiplicative constant. Since  $\|u\|_{L^p(\mu)}$  and  $\|u'\|_{L^p(\mu)}$  are bounded, we only have to consider the case where  $\|u - u'\|_{L^1(\mu)}$  is small. Define for a constant  $M$  large enough  $u_M := \max(u, -M)$  and  $u'_M := \max(u', -M)$ . Using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned} \|u - u'\|_{L^p(\mu)} &\lesssim \|u_M - u'_M\|_{L^p(\mu)} + e^{-\alpha M} \lesssim M^{(p-1)/p} \|u_M - u'_M\|_{L^1(\mu)}^{1/p} + e^{-\alpha M} \\ &\lesssim M^{(p-1)/p} \|u - u'\|_{L^1(\mu)}^{1/p} + M^{(p-1)/p} e^{-\alpha M/p} + e^{-\alpha M}. \end{aligned}$$

It is enough to choose  $M$  equal to a large constant times  $-\log \|u - u'\|_{L^1(\mu)}$ .  $\square$

Recall the following notion of capacity which was introduced by Kolodziej [27] and is related to the well-known Bedford-Taylor capacity [2]. For any Borel subset  $A$  of  $X$ , define

$$\text{cap}_{\text{BTK}}(A) := \sup \left\{ \int_A (dd^c u + \omega)^n, \quad u \text{ } \omega\text{-psh such that } 0 \leq u \leq 1 \right\}.$$

The following definition is inspired by the works of Kolodziej [26, 27, 28].

**Definition 2.3.** A positive measure  $\mu$  on  $X$  is said to be *K-moderate* if there are constants  $c > 0$  and  $\alpha > 0$  such that for every Borel subset  $A$  of  $X$  we have

$$\mu(A) \leq c \exp(-\text{cap}_{\text{BTK}}(A)^{-\alpha}).$$

We will also need the following notion of capacity introduced by Sibony and the first author [16] which is related to the capacities of Alexander [1] and of Siciak [30], see also Harvey-Lawson [24]. For any Borel subset  $A$  of  $X$ , define

$$\mathcal{T}(A) := \inf \left\{ \exp(\sup_A u), \quad u \text{ } \omega\text{-psh and } \max_X u = 0 \right\}.$$

Recall from Guedj-Zeriahi [23, Prop. 7.1] the following relation between  $\text{cap}_{\text{BTK}}(A)$  and  $\mathcal{T}(A)$  for every compact set  $A \subset X$ :

$$c_1 \exp(-\lambda_1 \text{cap}_{\text{BTK}}(A)^{-1}) \leq \mathcal{T}(A) \leq c_2 \exp(-\lambda_2 \text{cap}_{\text{BTK}}(A)^{-1/n}), \quad (1)$$

where  $c_i > 0$  and  $\lambda_i > 0$  are constants independent of  $A$ .

**Proposition 2.4.** *Let  $\mu$  be a probability measure on  $X$ . Then  $\mu$  is K-moderate if and only if it is weakly moderate, i.e., there are constants  $\lambda > 0$  and  $\alpha > 0$  such that*

$$\int \exp(\lambda |u|^\alpha) d\mu \leq c \quad \text{for } u \text{ } \omega\text{-psh with } \max_X u = 0.$$

*In particular, if  $\mu$  is moderate, then it is K-moderate.*

*Proof.* Assume that  $\mu$  is weakly moderate as above. We will show that it is K-moderate. It is enough to obtain the estimate in Definition 2.3 for every compact set  $A$  such that  $\text{cap}_{\text{BTK}}(A)$  is small enough. We can also assume that  $\text{cap}_{\text{BTK}}(A)$  is strictly positive since otherwise  $A$  is pluripolar and the fact that  $\mu$  is weakly moderate implies that  $\mu(A) = 0$ . So we also have  $\mathcal{T}(A) > 0$ .

Let  $u$  be an  $\omega$ -psh function with  $\max_X u = 0$  such that

$$u \leq \log \mathcal{T}(A) + 1 \text{ on } A.$$

Using that  $\mathcal{T}(A)$  is small, we obtain that

$$\mu(A) \leq \mu\{u \leq \log \mathcal{T}(A) + 1\} \leq \mu\{\lambda |u|^\alpha \geq (-\log \mathcal{T}(A))^{\alpha/2}\}.$$

Since  $\mu$  is weakly moderate, if  $c$  is the constant as in the lemma, the last quantity is bounded above by

$$c \exp \left( - (-\log \mathcal{T}(A))^{\alpha/2} \right),$$

which is, by the second inequality in (1), dominated by

$$c' \exp \left( - \text{cap}_{\text{BTK}}(A)^{-\alpha'} \right)$$

for some constants  $c' > 0$  and  $\alpha' > 0$ . So  $\mu$  is K-moderate.

Assume now that  $\mu$  is K-moderate as in Definition 2.3. We will show that it is weakly moderate. Let  $u$  be an  $\omega$ -psh function such that  $\max_X u = 0$ . It is enough to show for any  $M \geq 0$  that

$$\mu\{u < -M\} \leq c'' \exp(-M^{\alpha''})$$

for some constants  $c'' > 0$  and  $\alpha'' > 0$  independent of  $M$  and of  $u$ . Let  $A$  be an arbitrary compact subset of the open set  $\{u < -M\}$ . We want to bound  $\mu(A)$ . We have  $\mathcal{T}(A) \leq e^{-M}$  by definition of  $\mathcal{T}$ . We can assume that  $\text{cap}_{\text{BTK}}(A)$  and  $\mathcal{T}(A)$  are small enough. Since  $\mu$  is K-moderate, we obtain for some constant  $\alpha'' > 0$

$$\log \mu(A) \leq \text{const} - \text{cap}_{\text{BTK}}(A)^{-\alpha} \leq -(-\log \mathcal{T}(A))^{\alpha''} \leq -M^{\alpha''},$$

where the second inequality follows from the first estimate in (1). Hence,  $\mu$  is weakly moderate. The lemma follows.  $\square$

**Example 2.5.** Let  $\mu$  be a probability measure on  $\mathbb{P}^1$  smooth except at 0 and such that

$$\mu = |z|^{-2} \exp \left( - (-\log |z|)^{1/2} \right) (idz \wedge d\bar{z}) \quad \text{near } 0.$$

If  $u$  is equal to  $\log |z|$  near 0, we see that  $\exp(-\lambda u)$  is not  $\mu$ -integrable for any  $\lambda > 0$ . So  $\mu$  is not moderate. One can check that  $\mu$  is, however, weakly moderate and hence it is K-moderate.

### 3 Super-potentials of positive measures

The notion of super-potential was introduced by Sibony and the first author. It extends the notion of potential of positive closed  $(1, 1)$ -currents to positive closed  $(p, p)$ -currents and allows to solve some problems in complex dynamics. We recall it in the case of measures, i.e. for  $p = k$ .

Let  $\mathcal{C}$  denote the set of positive closed  $(1, 1)$ -currents in the cohomology class  $\{\omega\}$ . This is a convex compact set. We can consider for each real number  $\alpha > 0$  the following distance on  $\mathcal{C}$ :

$$\text{dist}_\alpha(T, T') := \sup_{\|\Phi\|_{\mathcal{C}^\alpha} \leq 1} |\langle T - T', \Phi \rangle|$$

where  $\Phi$  is a test smooth  $(n-1, n-1)$ -form. Observe that the family  $\text{dist}_\alpha$  is decreasing in  $\alpha$ . The following proposition was obtained in [17] as a consequence of the interpolation theory for Banach spaces.

**Proposition 3.1.** *Let  $\alpha$  and  $\beta$  be real numbers such that  $\beta \geq \alpha > 0$ . Then there is a constant  $c = c(\alpha, \beta) > 0$  such that*

$$\text{dist}_\beta \leq \text{dist}_\alpha \leq c(\text{dist}_\beta)^{\alpha/\beta}.$$

For each current  $T$  in  $\mathcal{C}$  there is a unique  $\omega$ -psh function  $u$  such that

$$T = dd^c u + \omega \quad \text{and} \quad \int_X u \omega^n = 0.$$

We call  $u$  the  $\omega$ -potential of  $T$ . For each real number  $p > 1$ , define the following distance on  $\mathcal{C}$

$$\text{dist}_{L^p}(T, T') := \|u - u'\|_{L^p},$$

where  $u$  and  $u'$  are  $\omega$ -potentials of  $T$  and  $T'$  respectively and the  $L^p$  norm is with respect to the measure  $\omega^n$ . Since we assume that  $\omega^n$  is a probability measure, the family  $\text{dist}_{L^p}$  is increasing in  $p$ . Note that  $\mathcal{C}$  has finite diameter with respect to all the above distances.

**Proposition 3.2.** *Let  $p \geq 1$  be a real number. Then there are constants  $c > 0$ ,  $c' > 0$ , and  $c'' > 0$  depending on  $p$  such that*

$$c \text{dist}_2 \leq \text{dist}_{L^1} \leq c' \text{dist}_1 \quad \text{and} \quad \text{dist}_{L^p} \leq c'' \max(1, -\log \text{dist}_{L^1})^{\frac{p-1}{p}} (\text{dist}_{L^1})^{\frac{1}{p}}.$$

*Proof.* Given  $T, T' \in \mathcal{C}$ , let  $u$  and  $u'$  be the  $\omega$ -potentials of  $T$  and  $T'$  respectively. Let  $\Phi$  be a smooth  $(n-1, n-1)$ -form such that  $\|\Phi\|_{\mathcal{C}^2} \leq 1$ . We have

$$|\langle T - T', \Phi \rangle| = |\langle u - u', dd^c \Phi \rangle| \lesssim \text{dist}_{L^1}(T, T').$$

So the first inequality in the proposition is clear.

Consider the second inequality in the proposition. Let  $\pi_1$  and  $\pi_2$  denote the canonical projections from  $X \times X$  onto its factors. Let  $\Delta$  be the diagonal of  $X \times X$ . In the proof of [15, Prop. 2.1], an explicit kernel  $K(x, y)$  on  $X \times X$  was obtained, see also Bost-Gillet-Soulé [3]. It is an  $(n-1, n-1)$ -form smooth outside  $\Delta$  such that

$$\|K(x, y)\| \lesssim -\log \text{dist}(x, y) \text{dist}(x, y)^{2-2n}$$

and

$$\|\nabla K(x, y)\| \lesssim \text{dist}(x, y)^{1-2n}$$

when  $(x, y)$  tends to  $\Delta$ . Here,  $\|\nabla K(x, y)\|$  denotes the sum of the norms of the gradients of the coefficients of  $K(x, y)$  for a fixed atlas on  $X \times X$ .

This kernel gives a solution  $v$  to the equation  $dd^c v = T - T'$  in the sense of currents with

$$v(x) := \int_{y \in X} K(x, y) \wedge (T(y) - T'(y))$$

or more formally

$$v := (\pi_1)_*(K \wedge \pi_2^*(T - T')).$$

Indeed, if  $[\Delta]$  denotes the current of integration on the diagonal  $\Delta$ , then  $[\Delta] - dd^c K$  is a smooth representation of the cohomology class of  $\Delta$  in the Künneth decomposition of  $H^{n,n}(X \times X, \mathbb{C})$ .

We show that  $\|v\|_{L^1} \lesssim \text{dist}_1(T, T')$ . Consider a test smooth  $(n, n)$ -form  $\Phi$  such that  $\|\Phi\|_\infty \leq 1$ . We have

$$\langle v, \Phi \rangle = \int_{X \times X} K(x, y) \wedge \Phi(x) \wedge (T(y) - T'(y)) = \langle T - T', (\pi_2)_*(K \wedge \pi_1^*(\Phi)) \rangle.$$

The estimates on  $K$  imply that  $(\pi_2)_*(K \wedge \pi_1^*(\Phi))$  is a form with bounded  $\mathcal{C}^1$  norm. We deduce that  $|\langle v, \Phi \rangle| \lesssim \text{dist}_1(T, T')$ . Since this property holds for every  $\Phi$ , we obtain that  $\|v\|_{L^1} \lesssim \text{dist}_1(T, T')$ .

Define  $m := \int_X v \omega^n$  and  $\tilde{v} := v - m$ . It follows from the above discussion that  $|m| \lesssim \text{dist}_1(T, T')$  and  $dd^c \tilde{v} = T - T' = dd^c(u - u')$ . Since the solution of the equation  $dd^c \tilde{v} = T - T'$  with  $\int_X \tilde{v} \omega^n = 0$  is unique, we obtain that  $\tilde{v} = u - u'$ . This, combined with the estimates  $\|v\|_{L^1} \lesssim \text{dist}_1(T, T')$  and  $|m| \lesssim \text{dist}_1(T, T')$ , implies that  $\|u - u'\|_{L^1} \lesssim \text{dist}_1(T, T')$ . The second inequality in the proposition follows.

Recall that the measure  $\omega^n$  is moderate, see [31, 36]. So Lemma 2.2 applied to  $\omega^n$  implies the last inequality in the proposition.  $\square$

Propositions 3.1 and 3.2 show that the distances considered above define the same topology on  $\mathcal{C}$ . It is not difficult to see that this topology is induced by the weak topology on currents. The above propositions also imply that the following notion does not depend on the choice of the distance on  $\mathcal{C}$  and therefore gives us a large flexibility to prove this Hölder property.

**Definition 3.3.** A function  $\mathcal{U} : \mathcal{C} \rightarrow \mathbb{R}$  is said to be *Hölder continuous* if it is Hölder continuous for one of the above distances on  $\mathcal{C}$ .

**Definition 3.4.** Let  $\mu$  be a positive measure on  $X$ . The *super-potential* of  $\mu$  is the function  $\mathcal{U} : \mathcal{C} \rightarrow \mathbb{R} \cup \{-\infty\}$  defined by

$$\mathcal{U}(T) := \int u d\mu$$

where  $u$  is the  $\omega$ -potential of  $T$ .



The infinite dimensional compact space  $\mathcal{C}$  admits some “complex structure” and the super-potential  $\mathcal{U}$  satisfies similar properties as the ones of quasi-psh functions in the finite dimensional case. However, we do not need these properties here. It is important to notice that the Hölder continuity, the continuity and the boundedness of the super-potential  $\mathcal{U}$  do not depend on the choice of the Kähler form  $\omega$  on  $X$ .

In what follows we study various sufficient conditions for a positive measure  $\mu$  to possess a Hölder continuous super-potential. We will need the following elementary lemma.

**Lemma 3.5.** *Let  $\mu$  be a positive measure on  $X$ . Then its super-potential  $\mathcal{U}$  is Hölder continuous with Hölder exponent  $0 < \beta \leq 1$  with respect to the distance  $\text{dist}_{L^1}$  on  $\mathcal{C}$  if and only if there is a constant  $c > 0$  such that*

$$\|u - u'\|_{L^1(\mu)} \leq c \max(\|u - u'\|_{L^1}, \|u - u'\|_{L^1}^\beta)$$

for all  $\omega$ -psh functions  $u$  and  $u'$ .

*Proof.* We first prove the necessary condition. Assume that  $\mathcal{U}$  is Hölder continuous with exponent  $\beta$  as above. Observe that it is enough to show that

$$\left| \int (u - u') d\mu \right| \lesssim \max(\|u - u'\|_{L^1}, \|u - u'\|_{L^1}^\beta).$$

Indeed, this inequality applied to  $u, \max(u, u')$  and then to  $u', \max(u, u')$  gives the result.

Consider first the case where  $\int_X u \omega^n = \int_X u' \omega^n = 0$ . Define  $T := dd^c u + \omega$  and  $T' := dd^c u' + \omega$ . Then, by hypothesis, we have

$$\left| \int (u - u') d\mu \right| = |\mathcal{U}(T) - \mathcal{U}(T')| \lesssim \text{dist}_{L^1}(T, T')^\beta = \|u - u'\|_{L^1}^\beta.$$

In the general case, define  $m := \int_X u \omega^n$  and  $m' := \int_X u' \omega^n$ . We can apply the first case to  $v := u - m$  and  $v' := u' - m'$ . In order to obtain the inequality in the lemma, it is enough to use the triangle inequality and to observe that  $|m - m'| \lesssim \|u - u'\|_{L^1}$ .

For the sufficient part, assume the inequality in the lemma. Consider two currents  $T$  and  $T'$  in  $\mathcal{C}$ . Denote by  $u$  and  $u'$  their  $\omega$ -potentials which belong to a fixed compact family of  $\omega$ -psh functions. Therefore,  $\|u\|_{L^1}$ ,  $\|u'\|_{L^1}$  and  $\|u - u'\|_{L^1}$  are bounded. The inequality in the lemma implies that

$$|\mathcal{U}(T) - \mathcal{U}(T')| = \left| \int (u - u') d\mu \right| \lesssim \|u - u'\|_{L^1}^\beta.$$

The lemma follows.  $\square$

**Proposition 3.6.** *Let  $\varphi$  be a Hölder continuous  $\omega$ -psh function on  $X$ . Then the measure  $\mu := (dd^c\varphi + \omega)^n$  has a Hölder continuous super-potential.*

*Proof.* Define  $\mu_k := (dd^c\varphi + \omega)^k \wedge \omega^{n-k}$ . We prove by induction on  $k$  that  $\mu_k$  has a Hölder continuous potential. Assume this is true for  $k-1$ . We will use the criterium given in Lemma 3.5. We can assume  $u \geq u'$  since we can always reduce the problem to the case with  $u, \max(u, u')$  and the case with  $u', \max(u, u')$ . Subtracting from  $u$  and  $u'$  a constant allows to assume that  $\int_X u\omega^n = 0$ .

Define also  $m' := \int_X u'\omega^n$  and  $\hat{u}' := u' - m'$ . So  $u$  and  $\hat{u}'$  belong to a fixed compact family of  $\omega$ -psh functions. We deduce that  $\|u\|_{L^1}$  and  $\|\hat{u}'\|_{L^1}$  are bounded. By [6, Ch.III (3.11)], the integrals  $\langle \mu, u \rangle$  and  $\langle \mu, \hat{u}' \rangle$  are also bounded. Therefore, by Lemma 3.5, we only have to consider the case where  $|m'|$  is bounded by a fixed constant large enough and to prove the inequality

$$\int (u - u') d\mu_k \lesssim \|u - u'\|_{L^1}^{\beta_k}$$

for some constant  $\beta_k > 0$ . Indeed,  $\|u - u'\|_{L^1}$  is bounded by a fixed constant.

Since  $\varphi$  is Hölder continuous, using a standard convolution and a partition of unity, we can write  $\varphi = \varphi_\epsilon + (\varphi - \varphi_\epsilon)$  with  $\|\varphi_\epsilon\|_{\mathcal{C}^2} \lesssim \epsilon^{-2}$  and  $|\varphi - \varphi_\epsilon| \lesssim \epsilon^\alpha$  for some  $\alpha > 0$ . We have for  $T := dd^c u + \omega$  and  $T' := dd^c u' + \omega$

$$\begin{aligned} |\langle \mu_k, u - u' \rangle| &\leq |\langle \mu_{k-1}, u - u' \rangle| + |\langle dd^c\varphi \wedge (dd^c\varphi + \omega)^{k-1} \wedge \omega^{n-k}, u - u' \rangle| \\ &\leq |\langle \mu_{k-1}, u - u' \rangle| + |\langle dd^c\varphi_\epsilon \wedge (dd^c\varphi + \omega)^{k-1} \wedge \omega^{n-k}, u - u' \rangle| \\ &\quad + |\langle (\varphi - \varphi_\epsilon) \wedge (dd^c\varphi + \omega)^{k-1} \wedge \omega^{n-k}, T - T' \rangle|. \end{aligned}$$

Since  $u - u' \geq 0$  and  $\pm dd^c\varphi_\epsilon$  are bounded by a constant times  $\epsilon^{-2}\omega$ , the second term in the last sum is bounded by a constant times  $\epsilon^{-2}|\langle \mu_{k-1}, u - u' \rangle|$ . Applying the Chern-Levine-Nirenberg inequality [6, Ch.III (3.3)] to the last term in the above sum, we see that this term is bounded by a constant times  $\epsilon^\alpha$ . This together with the induction hypothesis yields

$$|\langle \mu_k, u - u' \rangle| \lesssim \epsilon^{-2}|\langle \mu_{k-1}, u - u' \rangle| + \epsilon^\alpha \lesssim \epsilon^{-2}\|u - u'\|_{L^1}^{\beta_{k-1}} + \epsilon^\alpha$$

for some constant  $\beta_{k-1} > 0$ . Choosing  $\epsilon$  equal to a fixed constant small enough times  $\|u - u'\|_{L^1}^{\frac{\beta_{k-1}}{2+\alpha}}$ , the proof is thereby complete.

Note that we can show in the same way that a wedge-product of positive closed currents (of arbitrary bidegree) with Hölder continuous super-potential admits a Hölder continuous super-potential.  $\square$

The following result, together with Theorem 1.1, implies Corollary 1.2.

**Proposition 3.7.** *Let  $\mu$  be a positive measure on  $X$ . Assume that locally we can write  $\mu = dd^c U + \partial V + \bar{\partial} W$  with Hölder continuous forms  $U, V, W$  of bidegrees  $(n-1, n-1)$ ,  $(n-1, n)$  and  $(n, n-1)$  respectively. Then  $\mu$  admits a Hölder*

continuous super-potential. Moreover, the hypothesis on  $\mu$  is satisfied when  $\mu$  belongs to the Sobolev space  $W^{2n/p-2+\epsilon,p}(X)$  or to the Besov space  $B_{\infty,\infty}^{\epsilon-2}(X)$  for some  $\epsilon > 0$  and  $p > 1$ .

*Proof.* Consider a coordinate ball  $\mathbb{B}$  in  $X$  and  $\chi$  a smooth positive function with compact support in  $\mathbb{B}$ . We can assume that  $\mu = dd^c U + \partial V + \bar{\partial} W$  as above on  $\mathbb{B}$ . Using a partition of unity, it is enough to show that  $\mu' := \chi(dd^c U + \partial V + \bar{\partial} W)$  has a Hölder continuous super-potential.

For  $0 < \epsilon \ll 1$ , using the standard convolution, we can write  $U = U_\epsilon + (U - U_\epsilon)$  with  $\|U_\epsilon\|_{\mathcal{C}^2} \lesssim \epsilon^{-2}$  and  $\|U - U_\epsilon\|_\infty \lesssim \epsilon^\alpha$  for some constant  $\alpha > 0$ . We obtain in the same way the regularizing forms  $V_\epsilon$  and  $W_\epsilon$  of  $V$  and  $W$  respectively. Using these estimates, we have

$$\begin{aligned} |\langle \mu', u - u' \rangle| &\leq |\langle dd^c U, \chi(u - u') \rangle| + |\langle \partial V, \chi(u - u') \rangle| + |\langle \bar{\partial} W, \chi(u - u') \rangle| \\ &\leq |\langle dd^c U_\epsilon, \chi(u - u') \rangle| + |\langle U - U_\epsilon, dd^c(\chi(u - u')) \rangle| \\ &\quad + |\langle \partial V_\epsilon, \chi(u - u') \rangle| + |\langle V - V_\epsilon, \partial(\chi(u - u')) \rangle| \\ &\quad + |\langle \bar{\partial} W_\epsilon, \chi(u - u') \rangle| + |\langle W - W_\epsilon, \bar{\partial}(\chi(u - u')) \rangle|. \end{aligned}$$

Recall that  $\|du\|_{L^1}$  and  $\|du'\|_{L^1}$  are bounded by a constant. These properties are consequences of classical properties of psh functions. We can also obtain them using the estimates on  $K$  and  $\nabla K$  given in Proposition 3.2. Consequently,  $dd^c(\chi(u - u'))$ ,  $\partial(\chi(u - u'))$  and  $\bar{\partial}(\chi(u - u'))$  have bounded mass. This discussion, combined with the above estimates on  $|\langle \mu', u - u' \rangle|$  and the above mentioned properties of  $U_\epsilon$ ,  $V_\epsilon$ ,  $W_\epsilon$ , implies that

$$|\langle \mu', u - u' \rangle| \lesssim \epsilon^{-2} \|u - u'\|_{L^1} + \epsilon^\alpha.$$

To obtain the first assertion in the proposition it is enough to take  $\epsilon := \|u - u'\|_{L^1}^{\frac{1}{2+\alpha}}$ .

For the second assertion, locally on a small ball  $\mathbb{B}$  in  $X$  with holomorphic coordinates  $z$ , if  $u$  is a solution of the Laplacian equation  $\Delta u = \mu$  and if  $U$  is a suitable constant times  $u(dd^c \|z\|^2)^{n-1}$ , then  $dd^c U = \mu$ . When  $\mu$  belongs to the Sobolev space  $W^{2n/p-2+\epsilon,p}(X)$ , the function  $u$  is in  $W^{2n/p+\epsilon,p}(\mathbb{B})$ , see [4, p.198] for  $W^{k,p}$  with  $k \in \mathbb{N}$  and [34, p.186 and p.230] for the interpolation and the duality which allow to consider the case of  $W^{k,p}$  with  $k \in \mathbb{R}$ . By Sobolev's embedding theorem,  $u$  is Hölder continuous, see e.g. [4, p.168]. When  $\mu$  is in the Besov space  $B_{\infty,\infty}^{\epsilon-2}(X)$ , the solution  $u$  is in  $B_{\infty,\infty}^\epsilon$  which is the Hölder space  $\mathcal{C}^\epsilon$ . The result follows.  $\square$

The following result and Theorem 1.1 imply the main result of Hiep in [25].

**Proposition 3.8.** *Let  $\mu$  be a positive measure on  $X$ . Assume that there are constants  $c > 0$  and  $\alpha > 0$  such that if  $B$  is a ball of radius  $r$  in  $X$  we have  $\mu(B) \leq cr^{2n-2+\alpha}$ . Then  $\mu$  has a Hölder continuous super-potential.*

*Proof.* Let  $T, T', u, u'$ , the super-potential  $\mathcal{U}$ , the kernel  $K$ , the function  $v$  and the constant  $m$  be given in Proposition 3.2 above. We have

$$\begin{aligned}\mathcal{U}(T) - \mathcal{U}(T') &= \langle \mu, u - u' \rangle = \langle \mu, v - m \rangle \\ &= \langle \mu, (\pi_1)_*(K \wedge \pi_2^*(T - T')) \rangle - m \\ &= \langle T - T', (\pi_2)_*(K \wedge \pi_1^*(\mu)) \rangle - m.\end{aligned}$$

Since we already have good estimates on  $m$ , it is enough to check that the form  $\Phi := (\pi_2)_*(K \wedge \pi_1^*(\mu))$  is Hölder continuous.

Let  $\mathbb{B}$  be a coordinate ball in  $X$  that we identify with the unit ball in  $\mathbb{C}^n$ . We will show that  $\Phi$  is Hölder continuous near the origin  $0 \in \mathbb{C}^n$ . Let  $\chi$  be a smooth function with compact support in  $\mathbb{B} \times \mathbb{B}$  and equal to 1 near the origin. We have

$$\Phi = (\pi_2)_*(\chi K \wedge \pi_1^*(\mu)) + (\pi_2)_*((1 - \chi)K \wedge \pi_1^*(\mu)).$$

Since the form  $(1 - \chi)K$  is smooth near  $X \times \{0\}$ , the last expression in the above identity defines a smooth form near 0. It remains to show that  $\Psi := (\pi_2)_*(\chi K \wedge \pi_1^*(\mu))$  is Hölder continuous near 0.

Observe that the coefficients of  $\Psi$  have the form

$$\Theta(x) := \int_{y \in \mathbb{B}} H(x, y) d\mu(y)$$

where  $H$  is a coefficient of  $\chi K$ . We use now the estimates on  $K$  and  $\nabla K$  given in Proposition 3.2. Consider two points  $x$  and  $x'$  in  $\mathbb{B}$  near 0 and denote by  $D$  the ball of center  $x$  and of radius  $\rho := 2\|x - x'\|^{1/(2n)}$ . We have

$$\Theta(x) - \Theta(x') = \int_D H(x, y) d\mu(y) - \int_D H(x', y) d\mu(y) + \int_{\mathbb{B} \setminus D} (H(x, y) - H(x', y)) d\mu(y).$$

The estimate on  $K$  and the hypothesis on  $\mu$  imply that the first two terms are bounded by a constant times  $|\log \rho| \rho^\alpha$ . The estimate on  $\nabla K$  implies that

$$|H(x, y) - H(x', y)| \lesssim \|x - x'\| \rho^{1-2n} \quad \text{for } y \notin D.$$

Therefore, the last integral in the above identity is bounded by a constant times  $\|x - x'\|^{1/(2n)}$ . We deduce that  $\Theta$  is a Hölder continuous function. The proposition follows.  $\square$

**Proposition 3.9.** *Let  $\mu$  be a positive measure on  $X$  with a Hölder continuous super-potential. Then  $\mu$  is moderate.*

*Proof.* Let  $u$  be an  $\omega$ -psh function with  $\int_X u \omega^n = 0$  and set  $u_M := \max(u, -M)$  for  $M > 1$  large enough. By Lemma 2.2 applied to the measure  $\omega^n$ , there is a constant  $\alpha > 0$  independent of  $u$  and  $M$  such that

$$\left| \int_X u_M \omega^n \right| = \left| \int_X (u_M - u) \omega^n \right| \lesssim e^{-\alpha M}.$$

Denote by  $\beta$  a Hölder exponent of  $\mathcal{U}$  with respect to the distance  $\text{dist}_{L^1}$  on  $\mathcal{C}$  with  $0 < \beta \leq 1$ . Since  $M$  is large enough, the estimate above shows that  $\|u - u_M\|_{L^1}$  is small. We deduce from Lemma 3.5 and the inequality  $u_M - u > 1$  on  $\{u < -M - 1\}$  that

$$\mu\{u < -M - 1\} \leq \left| \int (u - u_M) d\mu \right| \lesssim \|u - u_M\|_{L^1}^\beta \lesssim e^{-\alpha\beta M}.$$

Thus,  $\mu$  is moderate.  $\square$

The following corollary gives us a large family of measures with Hölder continuous super-potential. It shows that the restriction of such a measure to a Borel set gives also a measure with Hölder continuous super-potential. Note that by definition the set of measures with Hölder continuous super-potential is a convex cone. We then deduce from Theorem 1.1 analogous properties for Monge-Ampère measures with Hölder continuous potential which have been obtained in [9].

**Corollary 3.10.** *Let  $\mu$  be a positive measure with a Hölder continuous super-potential. If  $f$  is a positive function in  $L^p(\mu)$  with  $p > 1$ , then  $f\mu$  admits a Hölder continuous super-potential.*

*Proof.* Let  $q \geq 1$  be the real number such that  $p^{-1} + q^{-1} = 1$ . Let  $\mathcal{V}$  be the super-potential of  $f\mu$ . Consider two currents  $T, T'$  in  $\mathcal{C}$  and their  $\omega$ -potentials  $u, u'$ . We have

$$|\mathcal{V}(T) - \mathcal{V}(T')| = |\langle f\mu, u - u' \rangle| \leq \|f\|_{L^p(\mu)} \|u - u'\|_{L^q(\mu)}.$$

By Proposition 3.9 and Lemma 2.2, the last expression is bounded by a constant times  $\|u - u'\|_{L^1(\mu)}^{1/(2q)}$ . Lemma 3.5 implies the result.  $\square$

**End of the proof of Theorem 1.1.** The necessary condition follows from Proposition 3.6. To prove the sufficient part assume that the super-potential  $\mathcal{U}$  of  $\mu$  is Hölder continuous with Hölder exponent  $0 < \beta \leq 1$  with respect to the distance  $\text{dist}_{L^1}$  on  $\mathcal{C}$ . Propositions 3.9 and 2.4 imply that for every  $p > 0$  there is a constant  $c > 0$  such that

$$\mu(A) \leq c \text{cap}_{\text{BTK}}(A)^p \quad \text{for any compact set } A.$$

In other words,  $\mu$  satisfies the condition  $\mathcal{H}(\infty)$  in the sense of [20] (see also Definition 2.6 in [9]) which is stronger than condition (A) in [26]. The latter work ensures the existence of a bounded  $\omega$ -psh solution  $u$  to  $(dd^c u + \omega)^n = \mu$  with the normalization  $\inf_X u = 1$  (for the reader's convenience we use here the notation  $u$  instead of  $\varphi$  as in the reference [9] that we use now).

For  $\delta > 0$  small enough, let  $\rho_\delta u$  denote the regularization of  $u$  defined in [9, Sec. 2], see also [7, 8]. It satisfies

$$\|\rho_\delta u - u\|_{L^1} \leq c\delta^2 \quad \text{and} \quad dd^c \rho_\delta u \geq -c\omega$$

for some constant  $c \geq 1$ . Lemma 3.5 applied to the  $\omega$ -psh functions  $c^{-1}\rho_\delta u$  and  $c^{-1}u$  implies that

$$\int_X |\rho_\delta u - u| d\mu \lesssim \|\rho_\delta u - u\|_{L^1}^\beta \lesssim \delta^{2\beta}. \quad (2)$$

We are now able to apply Proposition 3.3 in [9] which shows that  $u$  is Hölder continuous.

Now, we follow along the same lines as those given in the proof of Theorem A in [9] in order to get an explicit Hölder exponent of  $u$  in terms of  $n$  and  $\beta$ . It is enough to make the following changes:

- After (3.1) in [9]: let  $q := 1$ ,  $0 < \alpha_1 < \frac{2\beta}{n+1}$  and choose  $\epsilon > 0$ ,  $\alpha$ ,  $\alpha_0$  such that  $\alpha_1 < \alpha < \alpha_0 < 2\beta - \alpha_0(n + \epsilon)$ . Take  $f \equiv 1$  and set  $g := 0$  on  $E_0$  and  $g := c$  elsewhere with a constant  $c \geq 1$  such that  $\|g\mu\| = \|\mu\|$ .
- As in [12, 27], we solve for continuous  $\omega$ -psh function  $v$

$$(dd^c v + \omega)^n = g\mu \quad \text{with} \quad \max(u - v) = \max(v - u).$$

- (3.2) in [9] becomes now (2) which implies  $\int_{E_0} d\mu \leq c_2 \delta^{2\beta - \alpha_0}$  with  $c_2 > 0$ . The last relation replaces the inequality  $\int_{E_0} f\omega^n \leq c_2 \delta^{\frac{2 - \alpha_0}{q}}$  in [9].

We obtain that  $u$  is Hölder continuous with exponent  $\alpha_1$  for all  $0 < \alpha_1 < \frac{2\beta}{n+1}$ .  $\square$

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T.-C. Dinh, UPMC Univ Paris 06, UMR 7586, Institut de Mathématiques de Jussieu,  
4 place Jussieu, F-75005 Paris, France.

`dinh@math.jussieu.fr`, <http://www.math.jussieu.fr/~dinh>

V.-A. Nguyen, Mathématique-Bâtiment 425, UMR 8628, Université Paris-Sud, 91405  
Orsay, France.

`VietAnh.Nguyen@math.u-psud.fr`, <http://www.math.u-psud.fr/~vietanh>